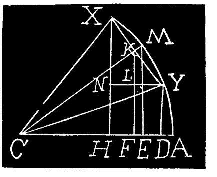
It is clear that $\frac{1}{n}$ of any magnitude may be repeated as a unit just as well as $\frac{n}{n}$ or $\frac{3n}{n}$; it is equally plain that $\frac{m}{n}$ is as much an expression of ratio as is m. Hence each definition applies to fractions as well as integers.

It is neither necessary nor advisable to divide ("break") single things (individuals, as apples) into parts in order to get fractions. In counting the eggs in a dozen (e. g.) the wee bairn is on the border of the fairyland of fractions, though he may not be conscious of it. At any stage of his counting the result is either integral or fractional. Five eggs is integral with respect to the *unit* (1 egg); it is fractional with respect to the *unity* or whole (dozen)—5 out of 12, 5 *twelfths*. Five *half-yards* is just as integral as 5 *yards*. The *ratio* in each is *five*. But in $\frac{5}{2}$ yards the ratio is $\frac{5}{2}$; the fractional idea is present, owing to the denominator, which defines the unit of measure.

SOME TRIGONOMETRIC RELATIONS PROVED GEOMETRICALLY.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

Most trigonometric formulæ may be proven geometrically in an elegant manner; and moreover, the relations between the trigonometric functions may be shown at a glance by means of the geometric figures. The results are all the more interesting, too, when proven also directly from first principles. For this reason the following exercises are offered.



For convenience, describe the arc AYX, and take the radius AC for the unit of measurement. Let the arc $AX=x$ and arc $AY=y$. Take M at the middle of XY, and draw lines as indicated.

Then $DY=\sin y$, $HX=\sin x$, $EM=\sin\frac{1}{2}(x+y)$, $KY=\sin\frac{1}{2}(x-y)$, $NX=\sin x-\sin y$, $NY=\cos y-\cos x$, $CE=\cos\frac{1}{2}(x+y)$, $CK=\cos\frac{1}{2}(x-y)$.

Now, $HX+DY=2KF=2EM\frac{KF}{EM}=2EM\frac{CK}{CM}=2EM.CK.$

That is, $\sin x+\sin y=2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$............(1).

by Drs. McClellan and Dewey. It is interesting in matter, vigorous and aggressive in style, refreshing in its originality, and scholarly in its conception and execution. It is in the 33d volume of the International Education Series, published by D. Appleton & Co., New York.

Again, $CH+CD=2CF=2CE.\frac{CF}{CE}=2CE\frac{CK}{CM}=2CE.CK,$

$$\text{or } \cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \dots\dots\dots\dots\dots (2).$$

The triangles CEM and XNY are similar;

$$\text{hence } \frac{NX}{NY}=\frac{CE}{CM}, \text{ or } NX=2CE\frac{\frac{1}{2}XY}{CM}=2CE.KY,$$

$$\text{that is, } \sin x-\sin y=\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \dots\dots\dots (3).$$

Similarly, $\frac{NY}{XY}=\frac{EM}{CM}$, or $NY=2EM\frac{\frac{1}{2}XY}{CM}=2EM.KY,$

$$\text{or } \cos x-\cos y=-2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \dots\dots\dots\dots\dots (4).$$

Equation (1) can be made very useful in computing trigonometric tables, as the writer intends subsequently to show.

Now let $AM=x$ and $MY=MX=y$. Then $AY=x-y$ and $AX=x+y$. We have $(CM)^2-(CK)^2=(CY)^2-(CK)^2=(KY)^2$. But $\frac{CM}{ME}=\frac{CK}{KF}=\frac{KY}{LY}$.

Therefore $(ME)^2-(KF)^2=(LY)^2=(KY)^2-(KL)^2$,

or $(KF)^2-(KL)^2=(ME)^2-(KY)^2$,

or $(KF+KL)(KF-KL)=(ME)^2-(KY)^2$,

or $HX\times DY=(ME)^2-(KY)^2$.

That is, $\sin(x+y)\sin(x-y)=\sin^2x-\sin^2y=\cos^2y-\cos^2x \dots\dots\dots\dots\dots (5).$

Again, $\frac{CM}{CE}=\frac{CK}{CF}=\frac{KY}{KL}$.

Therefore $(CE)^2-(CF)^2=(KL)^2=(KY)^2-(LY)^2$,

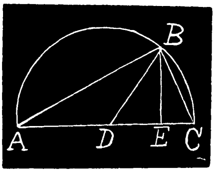
or $(CF)^2-(LY)^2=(CE)^2-(KY)^2$,

or $(CF-LY)(CF+LY)=CH\times CD=(CE)^2-(KY)^2$.

That is, $\cos(x+y)\cos(x-y)=\cos^2x-\sin^2y=\cos^2y-\sin^2x \dots\dots\dots\dots\dots (6).$

Let $DC=R$ the radius of a circle. Let the angle $CDB=2x$. Then $DAB=DBA=CBE=x$.

Then we have $\tan x=\frac{EC}{EB}$, also $\tan x=\frac{BE}{AE}$.



The product of these gives, $\tan^2x=\frac{CE}{AE}$, or $CE\times AE=(BE)^2$,

$$\text{or } \frac{EC}{AE}=\left(\frac{BE}{AE}\right)^2=\tan^2x.$$

Also, $\dfrac{EC}{BE}=\dfrac{\text{vers}2x}{\sin 2x}=\dfrac{1-\cos 2x}{\sin 2x}=\dfrac{\sin 2x}{1+\cos 2x}=\tan x$ [see above] ....(7).

Then $1+\tan^2 x=1+\dfrac{EC}{AE}=\dfrac{AC}{AE}=\dfrac{2R}{AE}$,

$$1-\tan^2 x=1-\frac{EC}{AE}=\frac{AE-EC}{AE}=\frac{AC-2EC}{AE}=\frac{2(R-EC)}{AE}.$$

$\cot 2x=\dfrac{DE}{BE}$ and $\operatorname{cosec} 2x=\dfrac{R}{BE}$. From these values we at once have,

$$\frac{2\tan x}{1+\tan^2 x}=\frac{2BE}{AE}\cdot\frac{AE}{2R}=\frac{BE}{R}=\sin 2x \text{ ....(8).}$$

$$\frac{2\tan x}{1-\tan^2 x}=\frac{2BE}{AE}\cdot\frac{AE}{2(R-EC)}=\frac{BE}{R-EC}=\frac{BE}{DE}=\tan 2x \text{ ....(9).}$$

$$\tan^2 x+2\cot 2x\tan x=\frac{EC}{AE}+\frac{2DE}{BE}\cdot\frac{BE}{AE}=\frac{EC+2DE}{AE}=\frac{AE}{AE}=1 \text{ ....(10).}$$

$$2\operatorname{cosec} 2x\tan x-\tan^2 x=\frac{2R}{BE}\cdot\frac{BE}{AE}-\frac{EC}{AE}=\frac{2R-EC}{AE}=\frac{AE}{AE}=1 \text{ ....(11).}$$

$$\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{2(R-EC)}{AE}\cdot\frac{AE}{2R}=\frac{R-CE}{R}=\frac{ED}{R}=\cos 2x \text{ ....(12).}$$

$$\operatorname{cosec} 2x-\cot 2x=\frac{R-ED}{BE}=\frac{EC}{BE}=\tan x=\frac{1-\cos 2x}{\sin 2x}=\frac{\sin 2x}{1+\cos 2x} \text{ ....(13).}$$

$$\operatorname{cosec} 2x+\cot 2x=\frac{R+ED}{BE}=\frac{AE}{BE}=\cot x=\frac{\sin 2x}{1-\cos 2x}=\frac{1+\cos 2x}{\sin 2x} \text{ ....(14).}$$

$$\frac{1+\sin 2x-\cos 2x}{1+\sin 2x+\cos 2x}=\frac{R+BE-ED}{R}\div\frac{R+BE+ED}{R}=\frac{EC+BE}{AE+BE}.$$

But $AE=\dfrac{(BE)^2}{EC}$; $\therefore\ \dfrac{CE+BE}{AE+BE}=\dfrac{EC+BE}{(BE)^2\div EC+BE}$

$$=\frac{EC(EC+BE)}{BE(EC+BE)}=\frac{EC}{BE}=\tan x \text{ ....(15).}$$

Again, $\cos x=\frac{AE}{AB}$, also $\cos x=\frac{AB}{AC}$.

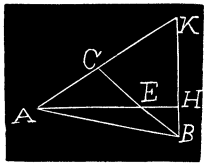
Twice the product of these gives $2\cos^2 x=\frac{2AE}{AC}=\frac{AE}{R}$.

Also $\cos 2x=\frac{DE}{R}$. $1+\cos 2x=\frac{DE+R}{R}=\frac{AE}{R}$. $\therefore 1+\cos 2x=2\cos^2 x$......(16).

$\sin x=\frac{CB}{AC}=\frac{BC}{2R}$; also $\sin x=\frac{EC}{BC}$. Twice the product of these gives

$2\sin^2 x=\frac{EC}{R}$. $1-\cos 2x=\frac{R-ED}{R}=\frac{EC}{R}$. $\therefore 1-\cos 2x=2\sin^2 x$..........(17).

To prove the "Tangent Proportion," let ABC be a plane triangle, the parts being represented as usual. Take $CE=CA$ and draw AEH. Draw BHK perpendicular to AH, to meet AC prolonged in K. Now considering the triangles ABC and ACE, the sum of the angles at A and E of the one is equal to the sum of the angles at A and B of the other. Hence $CAE+CEA=A+B$; and $CAE=CEA=BEH=\frac{1}{2}(A+B)$.



Also $BAE=A-\frac{1}{2}(A+B)=\frac{1}{2}(A-B)$. The angles at B and K of the triangle BCK are equal; for CBK is the complement of BEH or AEC, and BKC is the complement of the equal angle CAE. Hence $CK=CB=a$ and $AK=a+b$.

Now $\tan\frac{1}{2}(A-B)=\frac{BH}{AH}$ and $\tan\frac{1}{2}(A+B)=\frac{HK}{AH}$. $\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{BH}{HK}$.

But $\frac{BH}{HK}=\frac{BE}{AK}=\frac{a-b}{a+b}$. $\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{a-b}{a+b}$....................(1).

From the triangle ABE, $\frac{BE}{AB}=\frac{\sin BAE}{\sin AEC}$, or $\frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$..(2).

In the triangle AHK, $AH=AK\cos HAK=(a+b)\cos\frac{1}{2}(A+B)$.

In the triangle ABH, $AH=AB\cos BAH=c\cos\frac{1}{2}(A-B)$.

Equating, we have, $\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$.........................(3).

Equation (3) divided by (2) also gives (1).